Universidad Rey Juan Carlos
Escuela Técnica Superior de Ingeniería de Telecomunicación
Grado en Ingeniería Biomédica

MIDTERM EXAM II
Linear Algebra

Time: 110 minutes

- No electronic devices – including calculators –, books, or notes are allowed in the exam.
- All answers must be properly justified– otherwise they will not be considered.
- Answer exclusively to what you are being asked. Anything else that you add may be used against you.

Problem 1 Consider the linear map $T : \mathbb{P}_3 \to \mathbb{P}_4$ defined by $T[p(x)] = x\,p(x)$ for every $p(x) \in \mathbb{P}_3$.

(a) **(0.75 points)** Write the matrix of T with respect to the standard bases of $\mathbb{P}_3$ and $\mathbb{P}_4$.

(b) **(1.5 points)** Consider the polynomials $q_0(x) = 1$, $q_1(x) = 2x$, $q_2(x) = 4x^2 - 1$, $q_3(x) = 8x^3 - 4x$, and $q_4(x) = 16x^4 - 12x^2 + 1$. Compute the matrix of T with respect to the bases

$$\mathcal{B} = \{q_0(x),\, q_1(x), q_2(x), q_3(x)\} \quad \text{and} \quad \mathcal{C} = \mathcal{B} \cup \{q_4(x)\},$$

that is, $M_T^{\mathcal{C},\mathcal{B}}$.

(c) **(0.75 points)** Is T injective? Is T surjective? Justify your answers.

Problem 2 Let $T : \mathbb{R}^5 \to \mathbb{R}^5$. The matrix of T with respect to the standard basis is

$$M_T = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & -3 & 0 & 6 & 0 \\ 0 & 0 & -8 & 0 & 12 \\ 0 & 0 & 0 & -15 & 0 \\ 0 & 0 & 0 & 0 & -24 \end{pmatrix}$$

(a) **(1 point)** Without any calculations, explain why T is diagonalizable.

(b) **(1.5 points)** Diagonalize M_T.

Problem 3 **(2.5 points)** Let $\mathcal{E} = \{1, x, x^2, x^3, x^4\}$. Define the following inner product on $\mathbb{P}_4$:

$$\langle p(x), q(x) \rangle = [p(x)]_{\mathcal{E}}^t \begin{pmatrix} 1 & 0 & 1/4 & 0 & 1/8 \\ 0 & 1/4 & 0 & 1/8 & 0 \\ 1/4 & 0 & 1/8 & 0 & 5/64 \\ 0 & 1/8 & 0 & 5/64 & 0 \\ 1/8 & 0 & 5/64 & 0 & 7/128 \end{pmatrix} [q(x)]_{\mathcal{E}}, \qquad \forall p(x), q(x) \in \mathbb{P}_4.$$

Find a basis for $\mathbb{P}_4$ orthogonal with respect to $\langle\,,\rangle$.

Problem 4 **(2 points)** Find the best approximation in $\mathbb{P}_2$ with respect to the norm induced by $\langle\,,\rangle$ defined in the previous problem, of the polynomial $p(x) = x^4 - x^3 + \frac{1}{4}x^2 - \frac{1}{2}x + \frac{13}{16}$.
